NON-MEASURABLE SETS OF REALS WHOSE MEASURABLE SUBSETS ARE COUNTABLE

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ABSTRACT

In this paper we generalize Sierpinski's concept of sets of type S and give a characterization of such sets in terms of a partition of the reals. We also give a similar characterization of Lusin sets.

1. Introduction

In [2] Sierpinski considers sets of real numbers which have the property that every subset of (Lebesgue) measure zero is at most countable. Such a set is said to have *property S*. Since every set of positive measure contains a perfect set having measure zero, sets with property S can also be characterized as those sets of real numbers such that every uncountable subset is non-measurable. Sierpinski shows (p. 80) that if the continuum hypothesis is assumed, then there is an uncountable set having property S. In this paper we prove that if the continuum hypothesis holds, then there is a partition of the set of real numbers by an uncountable family of sets (each member of which is an uncountable set of measure zero) such that a subset of R has property S if and only if it intersects each member of the partition in a set which is at most countable. From this it is clear that there are uncountable sets which have property S. Since the only property of Lebesgue measure required to obtain this result is that it is non-atomic, we obtain the result for an arbitrary non-atomic Borel measure. In a final section, we apply the same methods to obtain a similar characterization of Lusin sets. In view of P. Cohen's proof of the independence of the continuum hypothesis from the axioms of set theory, it would be interesting to know whether or not the existence of sets with property S (or of Lusin sets) is also independent of these axioms. The present authors are not aware of any results in this direction.

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2. Property S

Let m be a fixed Borel measure on R. (Recall that every such measure is regular.) A set T in R is said to have *property S relative to m* if $T \cap E$ is at most countable for every set E of m-measure zero. By supp m denote the support of m (that is, the complement of the largest open set of m measure zero). The following is then obvious.

LEMMA 1. *Let T have property S relative to m. Then $T \cap (\text{supp } m)^c$ is at most countable.*

The measure m is called *atomic* if there is an (at most) countable set X such that $m(A) = \sum \{m(\{x\}) : x \in A \cap X\}$, for every Borel set A and $m(\{x\}) > 0$ for all $x \in X$. (In this case, we say that m is *concentrated* on X.) The measure m is called *diffuse* if $m(\{x\}) = 0$ for all $x \in R$. It is well known that if m is any Borel measure, there are unique Borel measures m_a and m_d such that m_a is atomic, m_d is diffuse and $m = m_a + m_d$.

LEMMA 2. *Let $m = m_a + m_d$ be the decomposition of m into its atomic and diffuse parts. A set T has property S relative to m if and only if it has property S relative to m_d.*

PROOF. ($\Leftarrow$) Let E have m-measure zero. Then E has m_d-measure zero and so $T \cap E$ is at most countable.

($\Rightarrow$) Let E have m_d-measure zero and let X be the set on which m_a is concentrated. Then $E - X$ has m_a-measure zero (and hence m-measure zero) so that $(T \cap E) - X$ is at most countable. But since X is at most countable, this implies $T \cap E$ is at most countable.

From the above lemma, the task of describing all sets having property S relative to m is equivalent to that of describing all sets having property S relative to m_d. Hence we will assume without loss of generality that m is diffuse. If m is the zero measure, it is clear that the family of sets having property S relative to m is exactly the family of all sets which are at most countable. So from now on we may assume that m is a non-zero diffuse measure. Thus, in particular, it follows that supp m is a perfect set (and hence uncountable).

Let $\mathscr{G}$ denote the set of all G_δ's in supp m which are uncountable and have m-measure zero. (By a G_δ in supp m is meant the intersection of an at most countable family of relatively open subsets of supp m or equivalently, the intersection of a G_δ of R with supp m.)

LEMMA 3. *The cardinal number of $\mathscr{G}$ is $2^{\aleph_0}$.*

PROOF. We will show that there is an uncountable G_δ set G in supp m having m-measure zero. Then for every finite set X in (supp $m - G$), $G \cup X$ is an uncountable G_δ in supp m having m-measure zero. Since there are $2^{\aleph_0}$ such sets disjoint from X, it follows that the cardinal number of $\mathscr{G}$ is at least $2^{\aleph_0}$. Since the cardinal number of the set of G_δ's in R is $2^{\aleph_0}$, the cardinal number of $\mathscr{G}$ is at most $2^{\aleph_0}$ and the result follows by the Schroeder-Bernstein theorem.

Thus we only need to show that there is an uncountable G_δ in supp m of m-measure zero. Without loss of generality, we may assume that $m(R) = 1$ and supp m is compact. Since m is diffuse, the distribution function f of m is continuous. (Recall that $f(t) = m((-\infty, t])$ for $t \in R$.) For $k = 1, 2, \cdots$ and for $i = 1, \cdots, 3^k$, define $t_{k,i} = \inf\{t : f(t) \geqq i/3^k\}$. Define $t_{k,0} = \sup\{t : f(t) = 0\}$. (Note that the set $W = \{t_{k,i}: k = 1, 2, \cdots \text{ and } i = 0, 1, \cdots, 3^k\}$ is a subset of supp m.)

For $k = 1, 2, \cdots$ define,

$$S_k = \cup\{(t_{k,3i+1}, t_{k,3i+2}) : i = 0, 1, \cdots, 3^{k-1}\}.$$

Then we have,

$$m(\cup\{S_k: k = 1, 2, \cdots, n\}) = 1 - (2/3)^{n+1}$$

Hence the set $G = \text{supp } m - \cup\{S_k: k = 1, 2, \cdots\}$ is a G_δ in supp m having m-measure zero. It is clear that every point in G is a limit point of the set $W \cap G$. Since G is a closed subset of R and $W \cap G \subset G$, it follows that G is a perfect subset of R and hence uncountable.

In order to characterize the family of sets having property S relative to m, we need some preliminary remarks. (We continue to assume that m is diffuse and not zero.) Assume the continuum hypothesis and enumerate the elements of supp m. Thus supp $m = \{x_\alpha: \alpha < \Omega\}$, where Ω is the first uncountable ordinal. Let $\mathscr{G} = \{E_\alpha: \alpha < \Omega\}$ be an enumeration of the uncountable G_δ's in supp m having m-measure zero. (This is possible by the above lemma.)

Let $\{A_\alpha : \alpha < \Omega\}$ be a family of subsets of supp m. For convenience we list here some properties (that the family may possess) which will be useful below.

1) For all $\alpha < \Omega$, the set $E_\alpha - \cup\{A_\beta: \beta \leqq \alpha\}$ is at most countable.

2) For all $\alpha < \Omega$, A_α is an uncountable set of m-measure zero.

3) For all $\alpha, \beta < \Omega$ $A_\alpha \cap A_\beta = \varnothing$.

4) For all $\alpha < \Omega$, $x_\alpha \in \cup\{A_\beta: \beta \leqq \alpha\}$.

The following theorem gives a characterization of subsets of R having property S relative to m in terms of the enumeration of $\mathscr{G}$ fixed above.

THEOREM 1. *Let $\{A_\alpha: \alpha < \Omega\}$ be a family of subsets of R which satisfies conditions 1 and 2 let T be a subset of R. Then T has property S relative to m if and only if $T \cap (\text{supp } m)^c$ and $T \cap A_\alpha$ are at most countable for all $\alpha < \Omega$.*

PROOF. ($\Rightarrow$) This is obvious.

($\Leftarrow$) Let E be an arbitrary G_δ of m-measure zero. Then $E \cap \text{supp } m = E_\alpha$ for some $\alpha < \Omega$. Thus,

$$T \cap E \subset [T \cap (\text{supp } m)^c] \cup [E_\alpha - \cup \{A_\beta: \beta \leqq \alpha\}] \cup [T \cap \{A_\beta: \beta \leqq \alpha\}].$$

Since all three sets on the right side of the above inequality are at most countable, it follows that $T \cap E$ is at most countable. Since the regularity of m implies that every set of m-measure zero is contained in a G_δ of m-measure zero, the proof is complete.

THEOREM 2. *There is a family $\{A_\alpha: \alpha < \Omega\}$ of subsets of R satisfying conditions 1, 2, 3, and 4 above.*

PROOF. Let $A_0 = E_0 \cup \{x_0\}$. Now assume that A_β has been defined for all $\beta < \alpha$ such that conditions 1, 2, 3 and 4 hold for the family $\{A_\beta: \beta < \alpha\}$. Define $S_\alpha = E_\alpha - \cup \{A_\beta: \beta < \alpha\}$. We consider two cases.

Case 1. If S_α is uncountable, define,

$$A_\alpha = S_\alpha \cup \{x_\alpha\} - \cup \{A_\beta: \beta < \alpha\}.$$

Case 2. If S_α is countable, let $A = \cup \{A_\beta: \beta < \alpha\}$. Then A has m-measure zero by Condition 2. Hence $R - A$ has positive m-measure. It then follows that $R - A$ contains a perfect G_δ set A^* of m-measure zero. (The proof is essentially that of Lemma 3 above.) Define,

$$A_\alpha = [A^* \cup \{x_\alpha\}] - \cup \{A_\beta: \beta < \alpha\}.$$

In either case, the family of sets satisfies conditions 1, 2, 3 and 4. The result then follows by transfinite induction.

The following consequences of Theorems 1 and 2 may now be derived. (Of course, we continue to assume the continuum hypothesis.)

THEOREM 3. *Let m be a Borel measure on R. There is a partition $\{A_\alpha: \alpha < \Omega\}$ of R by sets which are countable or of m-measure zero (and non-empty if m is non-atomic) such that a set T has property S relative to m if and only if $T \cap A_\alpha$ is at most countable for all $\alpha < \Omega$.*

PROOF. If m is atomic, let $A_0 = X =$ the set on which m is concentrated, let $A_1 = R - X$, and let $A_\alpha = \varnothing$ for $1 < \alpha < \Omega$. If m is not atomic apply Lemma 2 and Theorems 1 and 2.

Note in the case that m is diffuse, Theorem 2 actually guarantees the existence of a partition of R by uncountable sets of m-measure zero.

COROLLARY 1. *Let m be a non-atomic Borel measure. Then there is an uncountable set T having property S relative to m.*

PROOF. This is a consequence of Theorems 1 and 2.

COROLLARY 2. *Let m be a non-atomic Borel measure. Then there is an (uncountable) m-non-measurable set (in the sense of Carathéodory) with the property that every uncountable subset is also m-non-measurable.*

PROOF. Let T be an uncountable set having property S relative to m. If U is any uncountable subset of T which is measurable, then U must have positive m-measure. But then U contains an uncountable set with m-measure zero. (The proof is essentially the same as that of Lemma 3 above.) This is a contradiction.

COROLLARY 3. *Let m be a non-atomic Borel measure. A necessary and sufficient condition that a set Y contain an uncountable set having property S relative to m is that Y have positive exterior m-measure.*

PROOF. That the condition is necessary is obvious. In order to prove sufficiency, let $\{A_\alpha : \alpha < \Omega\}$ be a partition as in Theorem 3. Since Y is not an m-null set, the set $B = \{\alpha : Y \cap A_\alpha \neq \varnothing\}$ must be uncountable. By the axiom of choice, there is a set $T \subset Y$ such that $T \cap A_\alpha$ is a singleton for each $\alpha \in B$. Then T is an uncountable subset of Y and, by Theorem 3, satisfies property S relative to m.

In conclusion, we remark that Corollaries 1, 2 and 3 are generalizations of theorems proved by Sierpinski ([2], pp. 80, 87, 82) in the case that m is Lebesgue measure. We believe Theorems 1, 2 and 3 to be new. It is easy to show by example that Corollary 1 does not hold for an arbitrary partition.

3. Lusin sets

In this section, we apply the methods used to the study of Lusin sets. Recall that a set T is a *Lusin set* if the intersection of T with each perfect nowhere dense set is at most countable. It is known that the set $\mathscr{F}$ of perfect nowhere dense sets has cardinal number $2^{\aleph_0}$. Assuming the continuum hypothesis, let $\{x_\alpha : \alpha < \Omega\}$ and $\{F_\alpha : \alpha < \Omega\}$ be enumerations of R and $\mathscr{F}$ respectively.

THEOREM 4. *There is a partition $\{A_\alpha : \alpha < \Omega\}$ of R by non-empty nowhere dense sets such that T is a Lusin set if and only if $T \cap A_\alpha$ is at most countable for all $\alpha < \Omega$.*

PROOF. Define $A_0^* = F_0 \cup \{x_0\}$. If A_β^* has been defined for $\beta < \alpha$, define $A_\alpha^* = [F_\alpha \cup \{x_\alpha\}] - \cup \{A_\beta^* : \beta < \alpha\}$. Thus, by transfinite induction, A_α^* is defined for all $\alpha < \Omega$. Let $\mathscr{A} = \{A_\alpha^* : A_\alpha^* \neq \varnothing\}$. Since the sets belonging to $\mathscr{A}$ are nowhere dense and since $\cup \mathscr{A} = R$, Baire's category theorem implies that $\mathscr{A}$ is uncountable. Let $\mathscr{A} = \{A_\alpha : \alpha < \Omega\}$ be an enumeration of $\mathscr{A}$. Thus $\mathscr{A}$ is a partition of R by non-empty nowhere dense sets. Furthermore, since each set in $\mathscr{A}$ is a subset of a perfect nowhere set, it is clear that the condition of the theorem is necessary.

Let T be a set which intersects each member of $\mathscr{A}$ in a set which is at most countable and let F be any perfect nowhere sense set. Then $F = F_\alpha$ for some $\alpha < \Omega$. Thus,

$$T \cap F_\alpha \subseteq [T \cap A_\alpha^*] \cup [T \cap \cup \{A_\beta^* : \beta < \alpha\}].$$

Since $A_\gamma^* = \varnothing$ or $A_\gamma^* \in \mathscr{A}$ for all $\gamma < \Omega$, it follows that both sets on the right side of the above inequality are at most countable and hence that $T \cap F$ is at most countable.

COROLLARY 4. *There is an uncountable Lusin set.*

PROOF. This is immediate from Theorem 4.

REFERENCES

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